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The static theorem of limit analysis for masonry panels

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Constitutive equations

In order to characterize a masonry-like material we assume that the stress must be negative semidefinite and that the strain is the sum of two parts: the former depends linearly on the stress, the letter is orthogonal to the stress and positive semidefinite,

$$\boldsymbol{T} \in \mathrm{Sym}^{-}$$

$$\boldsymbol{E} = \boldsymbol{E}^{e} + \boldsymbol{E}^{f}$$

$$\boldsymbol{E}^{f} \in \mathrm{Sym}^{+}$$

$$\boldsymbol{T} = \mathbb{C}[\boldsymbol{E}^{e}]$$

$$\boldsymbol{E}^{f} \cdot \boldsymbol{T} = 0$$

where

$$\boldsymbol{E}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})$$

is the infinitesimal strain and $\mathbb{C}$ is symmetric and positive definite, i.e.,

$$\boldsymbol{A}_1 \cdot \mathbb{C}[\boldsymbol{A}_2] = \boldsymbol{A}_2 \cdot \mathbb{C}[\boldsymbol{A}_1],$$

$$\boldsymbol{A} \cdot \mathbb{C}[\boldsymbol{A}] > 0 \text{ for each } \boldsymbol{A} \in \text{Sym},\ \boldsymbol{A} \neq \boldsymbol{0}$$

Proposition

For each $\boldsymbol{E} \in \text{Sym}$ there exists a unique triplet

$(\boldsymbol{T}, \boldsymbol{E}^e, \boldsymbol{E}^f)$ that satisfies the constitutive equation.

Moreover, a masonry-like material is hyperelastic.

We write

$$w(\boldsymbol{E}) = \frac{1}{2}\boldsymbol{T}(\boldsymbol{E}) \cdot \boldsymbol{E}$$

for the stored energy.

Notations

Ω reference configuration of the body

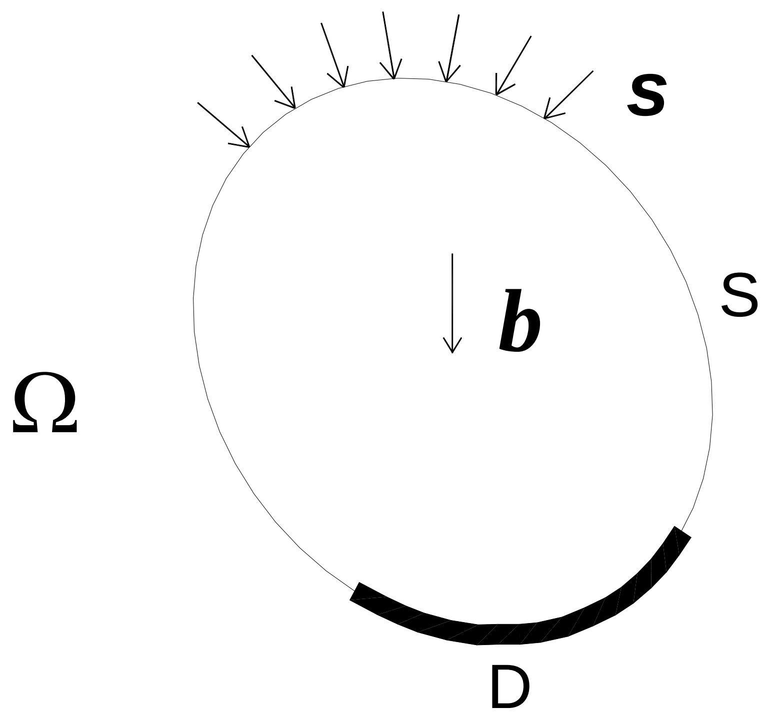
$$\partial\Omega = \mathcal{D} \cup \mathcal{S}, \qquad \text{such that } \mathcal{D} \cap \mathcal{S} = \emptyset$$

$\boldsymbol{u}$ displacement field, such that $\mathbf{u}_{|_{\mathcal{D}}} = \mathbf{0}$

$\boldsymbol{s} : \mathcal{S} \to \mathbb{R}^3$ surface traction

$\boldsymbol{b} : \Omega \to \mathbb{R}^3$ body force

$\boldsymbol{T} : \Omega \to \text{Sym}$ stress tensor



Limit analysis

The limit analysis deals with a family of loads $\mathcal{L}(\lambda)$ that depend linearly on a scalar parameter $\lambda \in \mathbb{R}$,

$$\mathcal{L}(\lambda) = (\boldsymbol{b}^{\lambda}, \boldsymbol{s}^{\lambda}) = (\boldsymbol{b}_0 + \lambda \boldsymbol{b}_1, \boldsymbol{s}_0 + \lambda \boldsymbol{s}_1)$$

$\boldsymbol{b}_0$, $\boldsymbol{s}_0$	permanent part of the load
$\boldsymbol{b}_1,\ \boldsymbol{s}_1$	variable part of the load
λ	loading multiplier

$\boldsymbol{b}_0$ and $\boldsymbol{b}_1$ are supposed to be square integrable functions on Ω, $\boldsymbol{s}_0$ and $\boldsymbol{s}_1$ are supposed to be square integrable functions on $\mathcal{S}$.

Let

$$V = \{\boldsymbol{v} \in W^{1,2}(\Omega, \mathbb{R}^3) : \boldsymbol{v} = \mathbf{0} \text{ a.e. on } \mathcal{D}\}$$

be the Sobolev space of all $\mathbb{R}^3$ valued maps such that $\boldsymbol{v}$ and the distributional derivative $\nabla\boldsymbol{v}$ of $\boldsymbol{v}$ are square integrable on Ω.

For each $(\lambda, \boldsymbol{v}) \in \mathbb{R} \times V$ we define the **potential energy** of the body

$$I(\lambda, \boldsymbol{v}) = \int_\Omega w(\boldsymbol{E}(\boldsymbol{v})dV - \int_\Omega \boldsymbol{v} \cdot \boldsymbol{b}^\lambda dV - \int_\mathcal{S} \boldsymbol{v} \cdot \boldsymbol{s}^\lambda dA$$

where

$$\int_\Omega w(\boldsymbol{E}(\boldsymbol{v})dV$$

is the strain energy and

$$< \mathcal{L}(\lambda), \boldsymbol{v} > \; = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{b}^\lambda dV + \int_\mathcal{S} \boldsymbol{v} \cdot \boldsymbol{s}^\lambda dA$$

is the work of the loads.

Moreover, we define the **infimum energy**

$$I_0(\lambda) = \inf\{I(\lambda, \boldsymbol{v}) : \boldsymbol{v} \in V\}.$$

Proposition

(i) The functional $I_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is **concave**,

$$I_0(a\lambda + (1-a)\mu) \geq aI_0(\lambda) + (1-a)I_0(\mu)$$

for every λ, $\mu \in \mathbb{R}$ and $a \in [0,1]$.

Therefore, the set

$$\Lambda := \{\lambda \in \mathbb{R} : I_0(\lambda) > -\infty\}$$

is an interval.

(ii) The functional I_0 is **upper semicontinuous**,

$$I_0(\lambda) \geq \limsup_{k\to\infty} I_0(\lambda_k)$$

for every $\lambda \in \mathbb{R}$ and every sequence $\lambda_k \to \lambda$.

We interpret the elements of Λ as loading multipliers for which the loads $\mathcal{L}(\lambda)$ are **safe**, i.e. the body does not collapse.

Each finite endpoint λ_c of the interval Λ is called a **collapse multiplier** with the interpretation that for $\lambda = \lambda_c$ or at least for λ arbitrarily close to λ_c outside Λ the body collapses.

We say that a stress field $\boldsymbol{T}$ is **admissible** if

$$\boldsymbol{T} \in L^2(\Omega, \mathrm{Sym}^-)$$

and that $\boldsymbol{T}$ **equilibrates the loads** $\mathcal{L}(\lambda) = (\boldsymbol{b}^\lambda, \boldsymbol{s}^\lambda)$ if

$$\int_\Omega \boldsymbol{T} \cdot \boldsymbol{E}(\boldsymbol{v}) dV = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{b}^\lambda dV + \int_\mathcal{S} \boldsymbol{v} \cdot \boldsymbol{s}^\lambda dA$$

for every $\boldsymbol{v} \in V$.

We say that the loads $\mathcal{L}(\lambda)$ are **compatible** if there exists a stress field $\boldsymbol{T}$ that is admissible and equilibrates $\mathcal{L}(\lambda)$.

Proposition (Static theorem of the limit analysis)

The loads $\mathcal{L}(\lambda)$ are compatible if and only if

$$I_0(\lambda) > -\infty.$$

That is the loads $\mathcal{L}(\lambda)$ are safe (i.e. $\lambda \in \Lambda$) if and only if

there exist a stress field $\boldsymbol{T}$ which

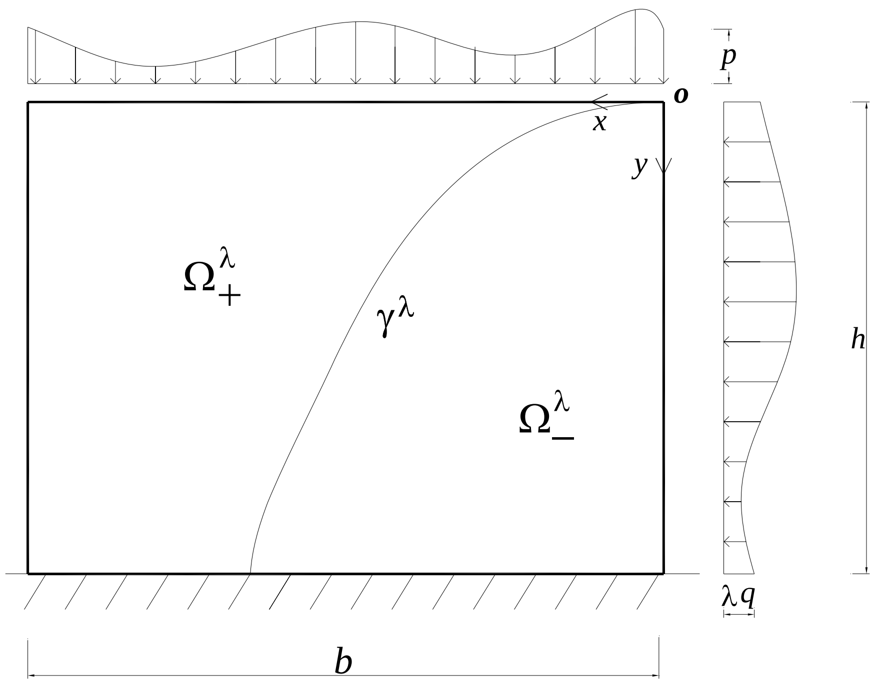
(i) is square integrable,

(ii) takes its values in Sym^-,

(iii) equilibrates the loads $\mathcal{L}(\lambda)$.

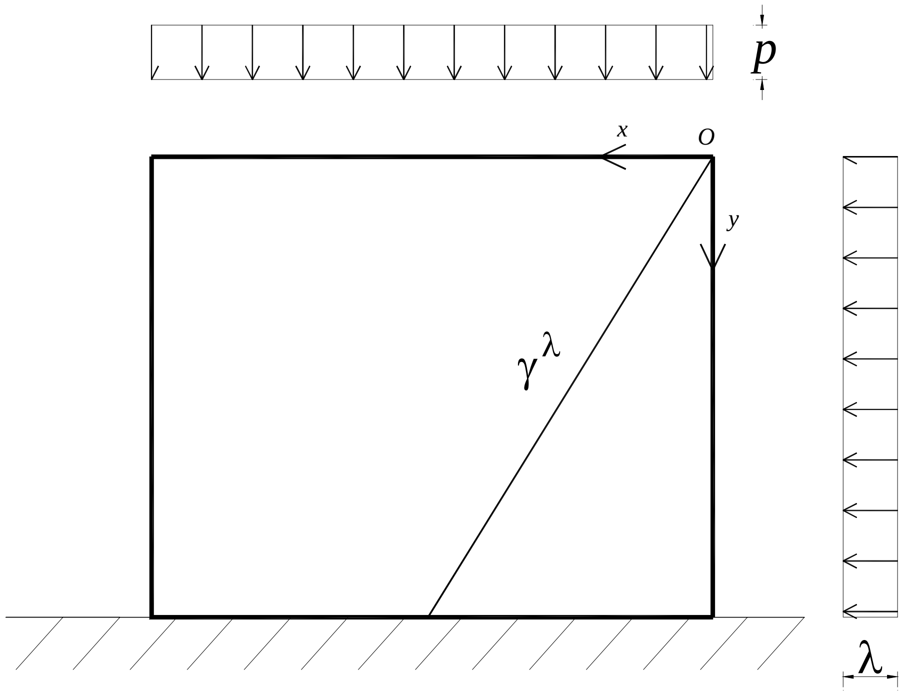
Rectangular panels

In the study of the static of masonry panels we verify that the problem of finding negative semidefinite stress fields that equilibrate the loads is considerably simplified if instead of stress fields represented by square integrable functions we allow the presence of curves of concentrated stress.



Example

In the following example, firstly, we will obtain a stress field which is a measure, whose divergence is a measure, that equilibrate the loads in a weak sense. Then we will see a procedure that, starting from this singular stress field, allow us to determine a stress field which is admissible and equilibrates the loads in the classical sense.



$$\Omega = (0,b) \times (0,h) \subset \mathbb{R}^2,$$

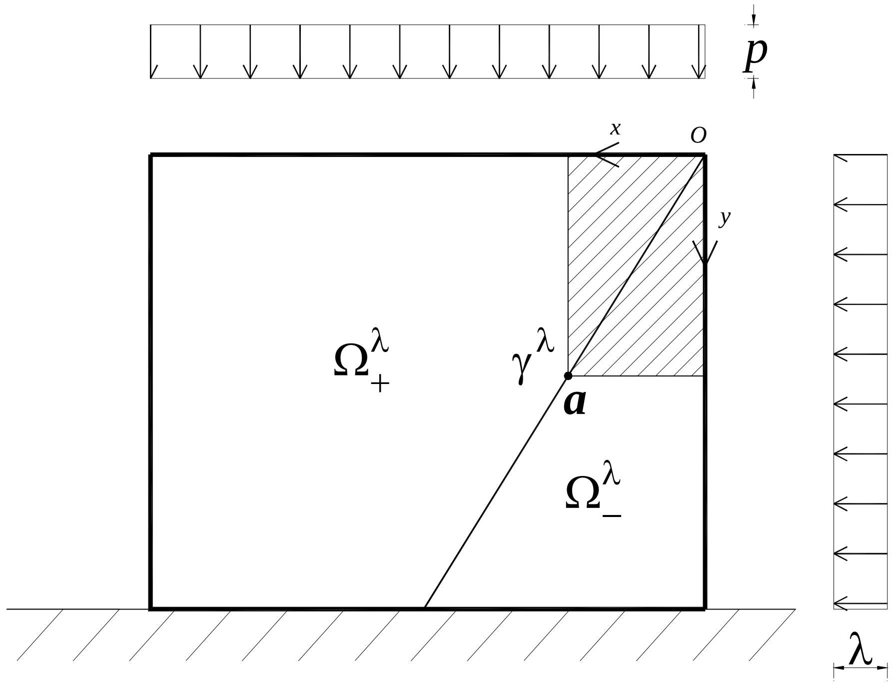
$$\mathcal{D} = (0,B) \times \{h\},$$

$$\mathcal{S} = \partial\Omega \backslash \mathcal{D},$$

$$\boldsymbol{b}^\lambda = \boldsymbol{0},$$

$$\boldsymbol{s}^\lambda(\boldsymbol{r}) = \begin{cases} p\boldsymbol{j}, \text{ on } (0,b) \times \{0\} \\ \lambda\boldsymbol{i}, \text{ on } \{0\} \times (0,h) \\ \boldsymbol{0} \quad \text{elsewhere} \end{cases}$$

where $p > 0, \;\; \lambda > 0$ and $\boldsymbol{r} = (x,y)$.



The singularity curve $\boldsymbol{\gamma}^{\lambda}$ can be obtained by imposing the equilibrium of the shaded rectangular region with the respect to the rotation about its left lower corner. We obtain

$$\boldsymbol{\gamma}^{\lambda} = \{(x, y) \in \Omega : y = \sqrt{\frac{p}{\lambda}}\, x\}$$

which divides Ω into the two regions $\Omega_{\pm}^{\lambda}$,

$$\Omega_{+}^{\lambda} = \{(x, y) \in \Omega : y > \sqrt{\frac{p}{\lambda}}\, x\},$$
$$\Omega_{-}^{\lambda} = \{(x, y) \in \Omega : y < \sqrt{\frac{p}{\lambda}}\, x\}.$$

The singularity curve $\boldsymbol{\gamma}^{\lambda}$ has to be wholly contained into the region Ω and this implies

$$0 < \lambda < \lambda_c$$

with

$$\lambda_c = \frac{pb^2}{h^2}.$$

For the regular part of the stress field $\boldsymbol{T}_r^\lambda$ (defined outside $\boldsymbol{\gamma}^\lambda$) we take

$$\boldsymbol{T}_r^\lambda(\boldsymbol{r}) = \begin{cases} -\,p\boldsymbol{j}\otimes\boldsymbol{j} & \text{if } \boldsymbol{r}\in\Omega_+^\lambda \\ -\,\lambda\boldsymbol{i}\otimes\boldsymbol{i} & \text{if } \boldsymbol{r}\in\Omega_-^\lambda. \end{cases}$$

The stress field defined on $\boldsymbol{\gamma}^\lambda$ can be obtained by imposing the equilibrium of the shaded rectangular region with the respect to the translation. We obtain

$$\boldsymbol{T}_s^\lambda(\boldsymbol{r}) = \,-\,\sqrt{p\lambda}\;\frac{\boldsymbol{r}\otimes\boldsymbol{r}}{|\boldsymbol{r}|}.$$

We note that $\dfrac{\boldsymbol{r}}{|\boldsymbol{r}|}$ is the unit tangent vector to $\boldsymbol{\gamma}^\lambda$.

The stress field

$$\mathbf{T}^{\lambda} = \boldsymbol{T}_r^{\lambda} + \boldsymbol{T}_s^{\lambda}$$

has to be interpreted as a **tensor valued measure**. That is, a function defined on the system of all Borel subsets of Ω which takes its values in Sym^{-} and is countably additive,

$$\mathbf{T}^{\lambda}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbf{T}^{\lambda}(A_i)$$

for each sequence of pairwise disjoint (borelian) sets.

$\mathbf{T}^{\lambda}$ is the sum of an absolutely continuous part (with respect to the area measure) with density $\boldsymbol{T}_r^{\lambda}$ and a singular, part concentrated on $\boldsymbol{\gamma}^{\lambda}$, with density $\boldsymbol{T}_s^{\lambda}$. Both the densities $\boldsymbol{T}_r^{\lambda}$ and $\boldsymbol{T}_s^{\lambda}$ are regular functions.

We can prove that the stress measure $\mathbf{T}^{\lambda}$ weakly equilibrates the loads $(\boldsymbol{b}^{\lambda}, \boldsymbol{s}^{\lambda})$, that is

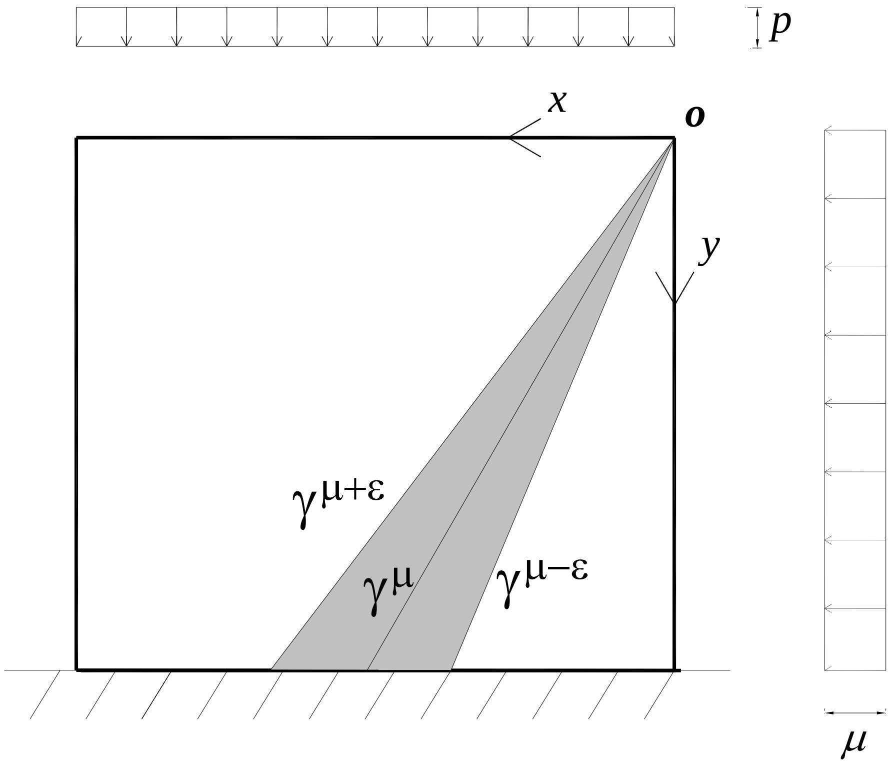
$$\int_{\Omega} \boldsymbol{E}(\boldsymbol{v}) d\mathbf{T} = \int_{\Omega} \boldsymbol{v} \cdot \boldsymbol{b}^{\lambda} dV + \int_{\mathcal{S}} \boldsymbol{v} \cdot \boldsymbol{s}^{\lambda} dA,$$

for any $\boldsymbol{v} \in V$.

Integration of measures

By the static theorem of limit analysis, in order to assert that the loads $\mathcal{L}(\lambda)$ are **compatible** we need an equilibrated stress field that is a square integrable function and then equilibrated stress measures are not enough.

Now we describe a procedure that in certain cases allows us to use the stress measure $\mathbf{T}^{\lambda}$ to determine a square integrable stress field $\boldsymbol{T}^{\lambda}$. Crucial to this procedure is the fact that both the loads $(\boldsymbol{s}^{\lambda},\ b^{\lambda})$ and the admissible equilibrating stress measure $\mathbf{T}^{\lambda}$ depend on a parameter λ.



The idea is to take the average of the stress measure over any set $(\mu - \epsilon, \mu + \epsilon)$, where $\epsilon > 0$ is sufficiently small. Averaging gives the measure

$$\mathbf{T} = \frac{1}{2\epsilon} \int_{\mu-\epsilon}^{\mu+\epsilon} \mathbf{T}^{\lambda} d\lambda.$$

It may happen that this measure, in contrast to $\mathbf{T}^{\mu}$, is absolutely continuous with respect to the area measure with a square integrable density.

For the previous example we obtain the following result.

If $0 < \lambda < \lambda_c$, then the loads $\mathcal{L}(\mu)$ are compatible. In fact if $(\mu - \epsilon, \mu + \epsilon) \subset (0, \lambda_c)$ then the measure

$$\mathbf{T} = \frac{1}{2\epsilon} \int_{\mu-\epsilon}^{\mu+\epsilon} \mathbf{T}^{\lambda} d\lambda$$

is absolutely continuous with respect to the area (Lebesgue) measure with density $\boldsymbol{T}$.

$\boldsymbol{T}$ is a bounded admissible stress field on Ω that equilibrates the loads $\mathcal{L}(\mu)$. We have

$$\boldsymbol{T} = \boldsymbol{T}_r + \boldsymbol{T}_s.$$

The densities $\boldsymbol{T}_r$ and $\boldsymbol{T}_s$ can be explicitly calculated. In fact, we have

$$\boldsymbol{T}_r(\boldsymbol{r}) = \frac{1}{2\epsilon}\int_{\mu-\epsilon}^{\mu+\epsilon} \boldsymbol{T}_r^\lambda(\boldsymbol{r})d\lambda \; =$$

$$\begin{cases} -p\boldsymbol{j}\otimes\boldsymbol{j} & \text{if } \boldsymbol{r}\in\Omega_+^\lambda\backslash A \\ -\mu\boldsymbol{i}\otimes\boldsymbol{i} & \text{if } \boldsymbol{r}\in\Omega_-^\lambda\backslash A \\ (2\epsilon)^{-1}(\alpha(\boldsymbol{r})\boldsymbol{i}\otimes\boldsymbol{i} \; + \beta(\boldsymbol{r})\boldsymbol{j}\otimes\boldsymbol{j}\,) & \text{if } \boldsymbol{r}\in A \end{cases}$$

where

$$A = \{(x,y)\in\Omega : \mu-\epsilon < px^2/y^2 < \mu+\epsilon\}$$

is the shaded area in the figure, and

$$\alpha(\boldsymbol{r}) = \frac{1}{2}(p^2x^4/y^4 - (\mu+\epsilon)^2),$$

$$\beta(\boldsymbol{r}) = p(\mu-\epsilon-p^2x^4/y^4).$$

Moreover,

$$\boldsymbol{T}_s(\boldsymbol{r}) = \begin{cases} \dfrac{-\,p^2x^2}{\epsilon y^4}\boldsymbol{r}\otimes\boldsymbol{r} & \text{if } \boldsymbol{r}\in A \\ 0 & \text{otherwise} \end{cases}$$

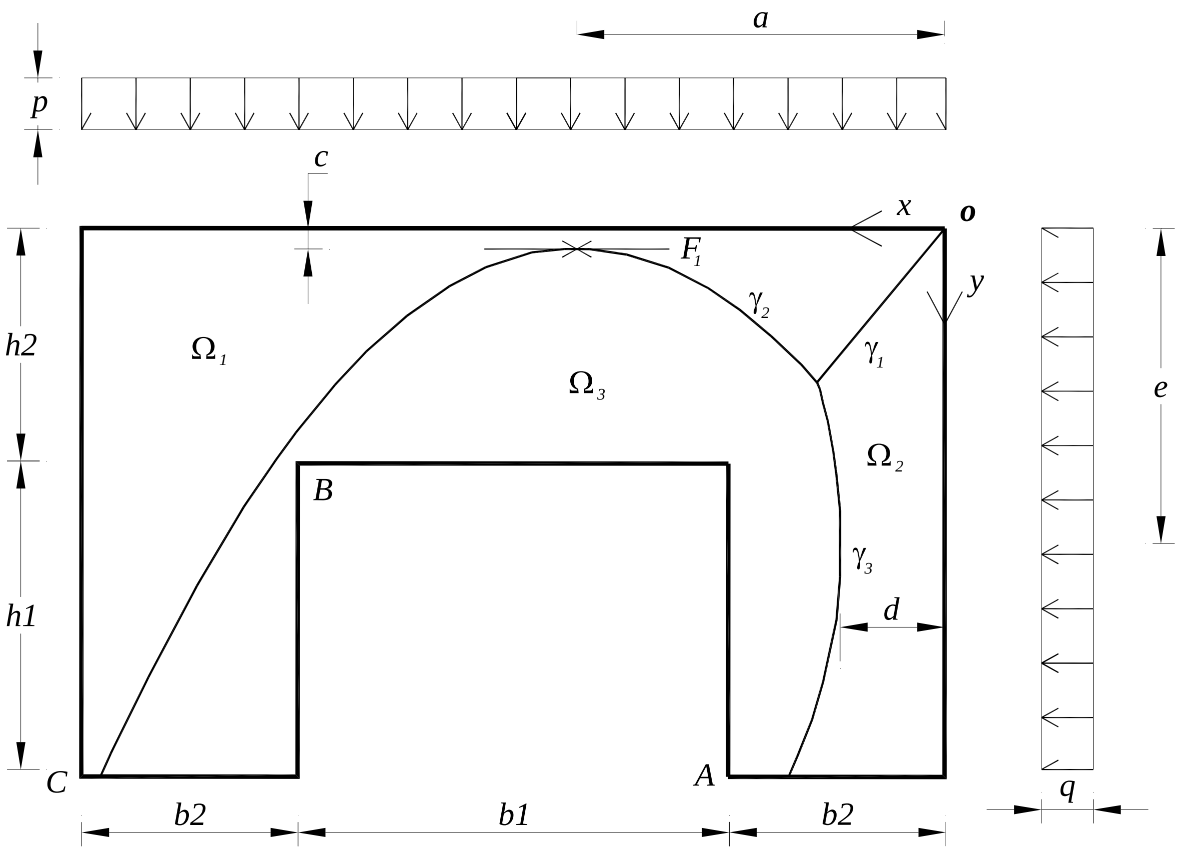
It is easily to verify that the stress field

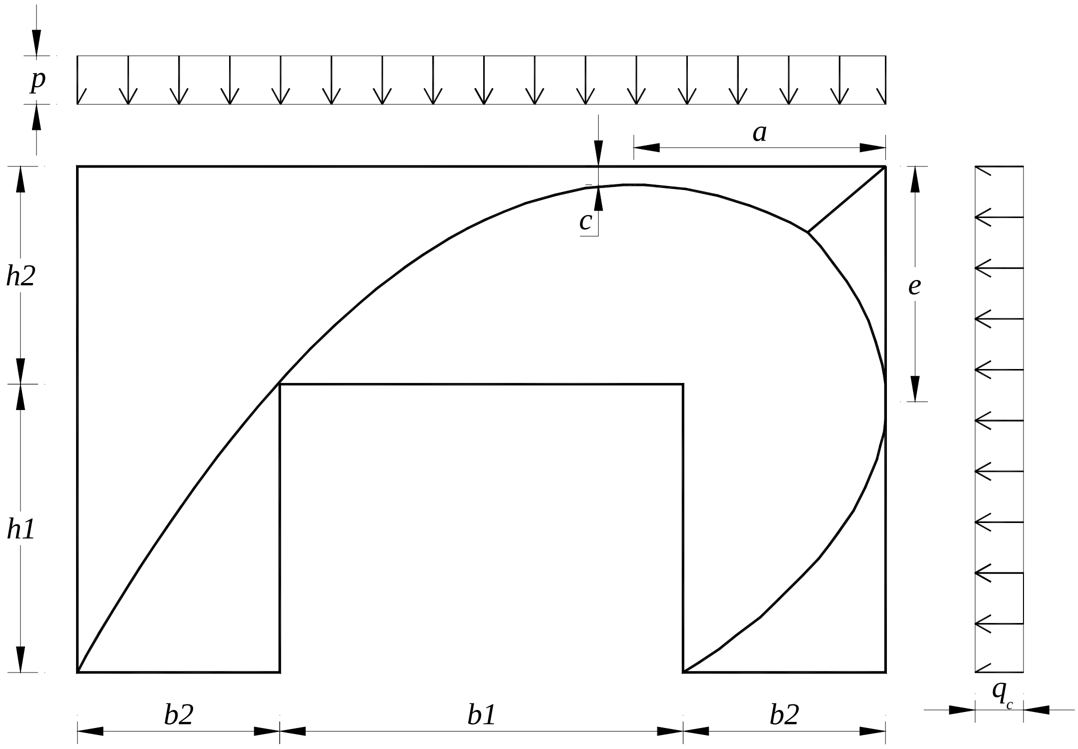
$$\boldsymbol{T} = \boldsymbol{T}_r + \boldsymbol{T}_s$$

is admissible, i.e. $\boldsymbol{T}$ a square integrable function which takes its values in Sym^-, and that it equilibrates the loads $\mathcal{L}(\mu)$,

$$\boldsymbol{T}\boldsymbol{n} = \boldsymbol{s}^\mu \text{ on } \mathcal{S}, \quad \text{div}\,\boldsymbol{T} = \boldsymbol{0} \text{ in } \Omega.$$

Panels with opening

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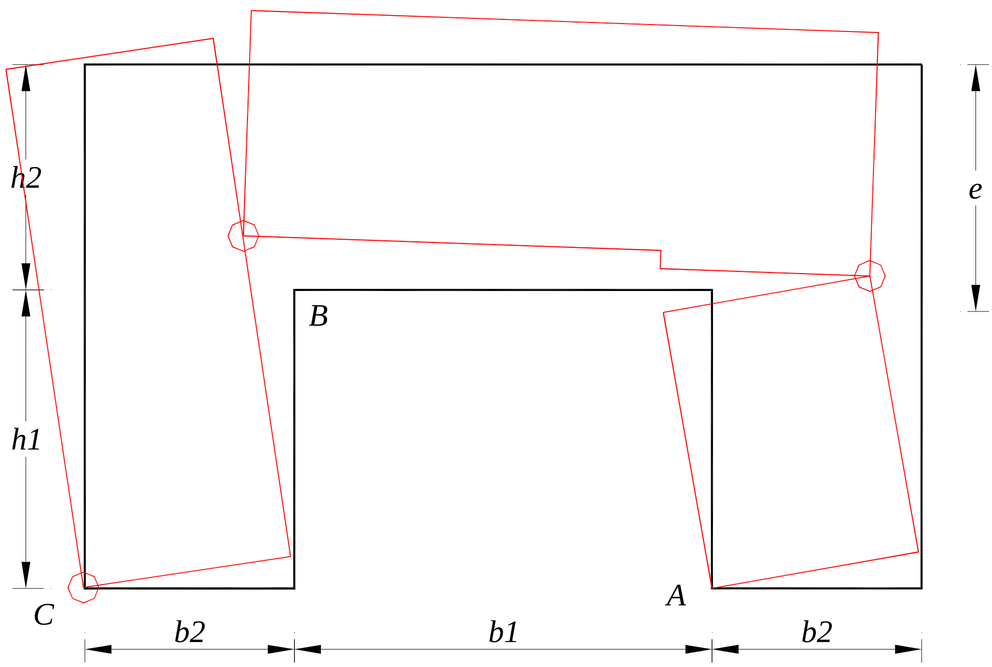
h2
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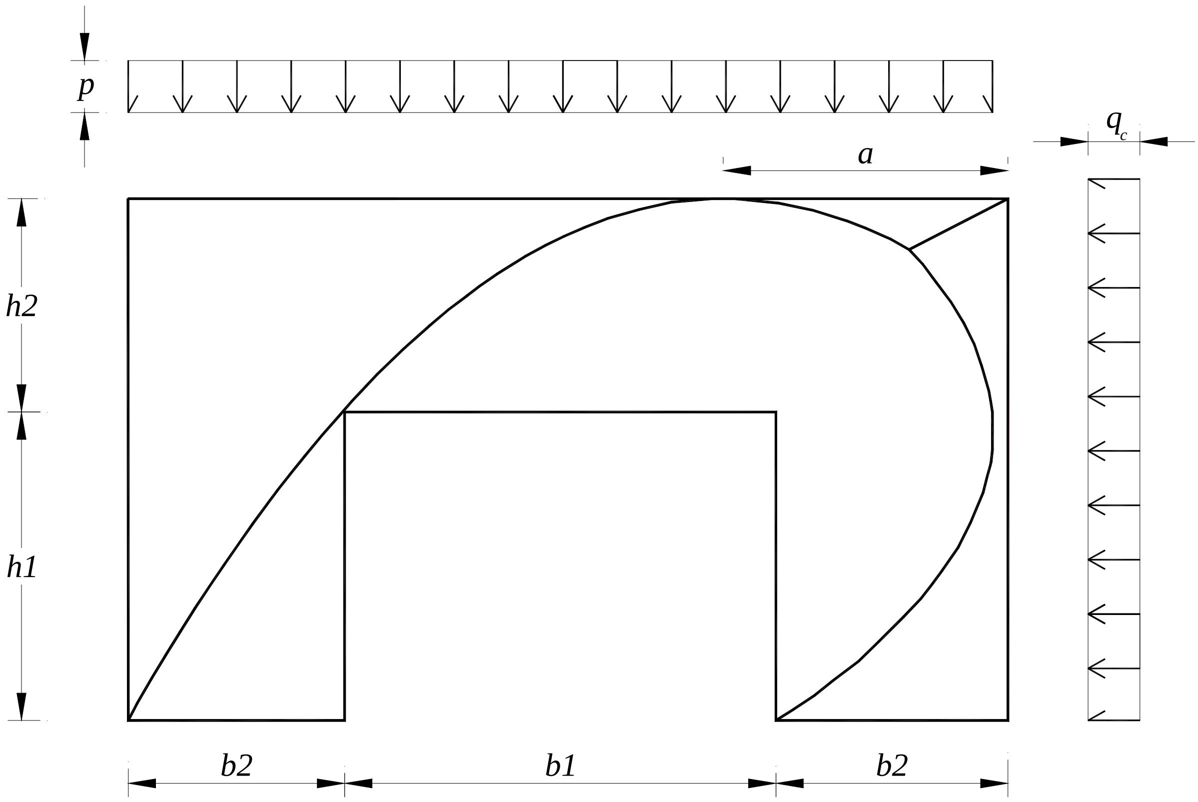
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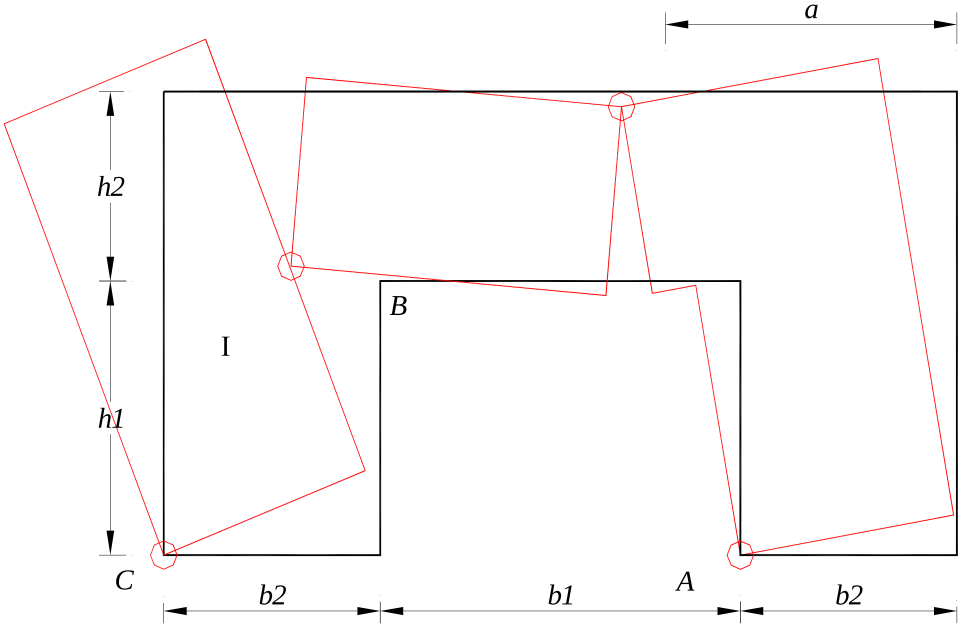
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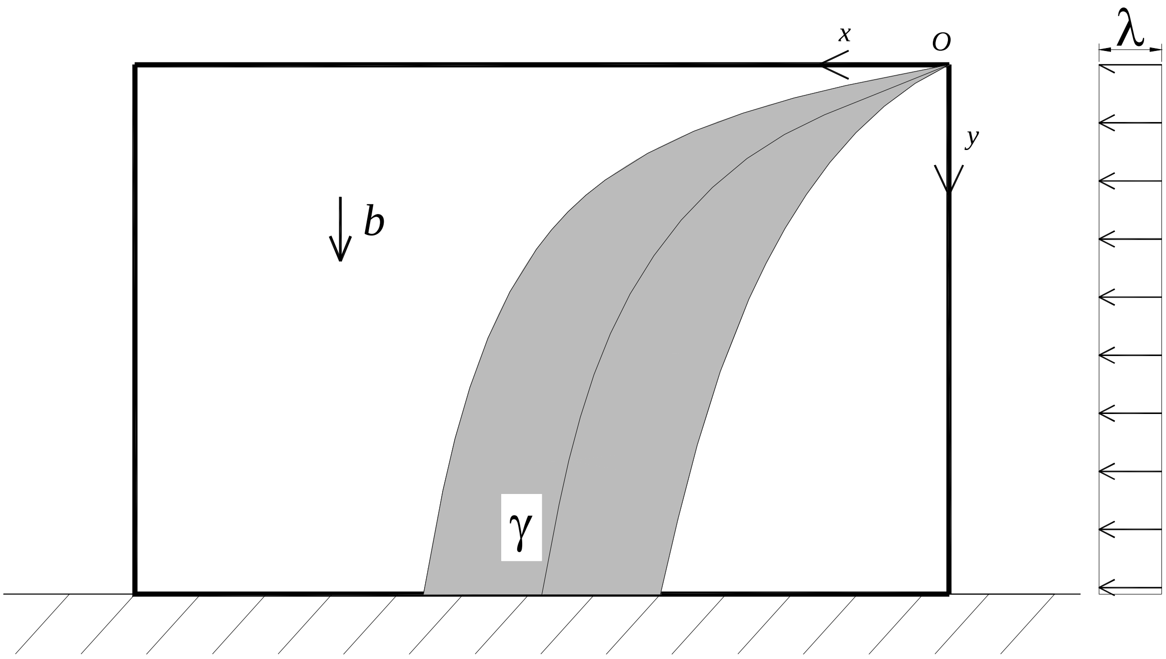
Panel under gravity



The panel is subjected to a side loads and its own gravity

$$\boldsymbol{s}^{\lambda}(\boldsymbol{r}) = \begin{cases} \lambda \boldsymbol{i} & \text{on } \{0\} \times (0, H), \\ \boldsymbol{0} & \text{elsewhere.} \end{cases}$$

The singularity curve is

$$\gamma^\lambda = \Big\{(x,y) \in \Omega : y = cbx^2/\lambda\Big\}, c = 1/2 + \sqrt{3}/6.$$

and

$$\lambda_c = cbB^2/H.$$

$$\boldsymbol{T}_r^\lambda(\boldsymbol{r}) = \begin{cases} -\, by\boldsymbol{j} \otimes \boldsymbol{j} & \text{in } \Omega_+^\lambda\ , \\ -\, \lambda\boldsymbol{i} \otimes \boldsymbol{i} - bx\boldsymbol{i} \odot \boldsymbol{j} - \dfrac{b^2x^2}{\lambda}\boldsymbol{j} \otimes \boldsymbol{j} & \text{in } \ \Omega_-^\lambda, \end{cases}$$

with $\boldsymbol{i} \odot \boldsymbol{j} = \boldsymbol{i} \otimes \boldsymbol{j} + \boldsymbol{j} \otimes \boldsymbol{i}$.

$$\boldsymbol{T}_s^\lambda = \sigma^\lambda \boldsymbol{t}^\lambda \otimes \boldsymbol{t}^\lambda,$$

with

$$\sigma^\lambda(\boldsymbol{r}) = -\frac{\sqrt{3}}{6} bx\sqrt{x^2 + 4y^2}$$

and

$$\boldsymbol{t}^\lambda(\boldsymbol{r}) = \frac{(x, 2y)}{\sqrt{x^2 + 4y^2}}$$

the unit tangent vector to $\boldsymbol{\gamma}^\lambda$.

By integration we obtain

$$\boldsymbol{T}(\boldsymbol{r}) = \begin{cases} -\,by\boldsymbol{j}\otimes\boldsymbol{j} & \text{in } \Omega^{\lambda}_{+}\backslash A\ , \\ -\,\mu\boldsymbol{i}\otimes\boldsymbol{i} - bx\boldsymbol{i}\odot\boldsymbol{j} - b^2x^2/\lambda\boldsymbol{j}\otimes\boldsymbol{j} & \text{in } \ \Omega^{\lambda}_{-}\backslash A, \\ \boldsymbol{S}(\boldsymbol{r}) & \text{in } A, \end{cases}$$

where

$$A = \{\boldsymbol{r} = (x,y) : bcx^2/y \in (\mu-\varepsilon, \mu+\varepsilon)\}$$

and

$$\boldsymbol{S}(\boldsymbol{r}) \ = \ -\,(2\varepsilon)^{-1}\Big\{\Big[-\frac{b^2x^4}{12y^2} + \frac{1}{2}(\mu+\epsilon)^2\Big]\boldsymbol{i}\otimes\boldsymbol{i} +$$

$$\Big[-\frac{b^2x^3}{3y} + bx(\mu+\varepsilon)\Big]\boldsymbol{i}\odot\boldsymbol{j} +$$

$$\Big[\Big(\frac{\sqrt{3}}{2} + \frac{5}{6} + ln\frac{y(\mu+\varepsilon)}{bcx^2}\Big)b^2x^2 - by(\mu-\epsilon)\Big]\boldsymbol{j}\otimes\boldsymbol{j}\Big\}.$$

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